NUMERICAL INTEGRATION RULES NEAR GAUSSIAN QUADRATURE†

BY

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ABSTRACT

We call a numerical integration formula based on k nodes which is exact for polynomials of degree at most n an (n, k) formula. Gaussian quadrature is the unique $(2k - 1, k)$ formula. In this paper we give a complete description of all $(2k - 3, k)$ formulas, including a characterization of those having all positive weights.

1. Introduction

Given a positive integer k and a weight function $\rho(t)$ defined on $I\colon [-1,1]$, if there exist nodes $x\colon (x_1,\cdots,x_k)$, $-1 \leqq x_1 < x_2 < \cdots < x_k \leqq 1$ and weights $w\colon(w_1,\cdots,w_k)$, $w_i \neq 0$, $i = 1,\cdots,k$, such that

$$(1)\qquad \int_{-1}^{1} p(t)\rho(t)dt = \sum_{i=1}^{k} w_i p(x_i),$$

for all $p \in P_n$ (where P_n is the set of polynomials of degree at most n) we say that (1) is an (n,k) quadrature formula (qf) based on x with weights w. Every formula (1) for $n \geqq k-1$ is interpolatory in the sense that if $L_k(f) \in P_{k-1}$ satisfies

$$L_k(f;x_i) = f(x_i),\ i = 1,\cdots,k$$

then

$$\sum_{i=1}^{k} w_i p(x_i) = \int_{-1}^{1} L_k(p;t)\rho(t)dt.$$

Since

$$L_k(p;t) = \sum_{i=1}^{k} p(x_i) l_i(x),$$

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and $n \geqq k-1$, w and x are related by

$$w_i = \int_{-1}^{1} l_i(t)\rho(t)\,dt. \tag{2}$$

There is a unique $(k-1,\ k)$ qf based on any x provided that the w_i determined by (2) are not equal to zero. At the other extreme, there is no (n,k) qf for $n \geqq 2k$ and there is exactly one $(2k-1,k)$ qf, based on the k zeros of p_k, the orthogonal polynomial of degree k with respect to ρ. This is called Gaussian quadrature. If w is strictly positive, that is, $w_i > 0$, $i = 1,\cdots,k$, we say that (1) is a positive (n,k) qf. Our main objective is to obtain a detailed description of $(2k-3,k)$ qf for $k \geqq 2$. Since every (n,k) qf is an (m,k) qf for $m < n$, we will easily be able to deduce a description of $(2k-2,k)$ qf.

The orthogonal polynomials $\{p_j\}$ with respect to ρ are important tools for us. We suppose them normalized so that the leading coefficient of each p_j is 1. The orthogonal polynomials satisfy the three term recurrence formula

$$p_j(t) = (t-u_j)p_{j-1}(t) - v_j p_{j-2}(t),\ j = 2,3,\cdots$$

where

$$p_0 = 1,\ p_1(t) = t - u_1$$

$$u_j = \left(\int_{-1}^{1} tp_{j-1}^2(t)\rho(t)dt\right) \Big/ \left(\int_{-1}^{1} p_{j-1}^2(t)\rho(t)dt\right),\ j = 1,2,\cdots,$$

and

$$v_j = \left(\int_{-1}^{1} p_{j-1}^2(t)\rho(t)dt\right) \Big/ \left(\int_{-1}^{1} p_{j-2}^2(t)\rho(t)\,dt\right),\ j = 2,3,\cdots.$$

Note that $v_j > 0$ and $|u_j| < 1$.

An easy argument, based on the orthogonality of the $\{p_j\}$, now shows that if (1) is an (n,k) qf (with $k-1 \leqq n \leqq 2k-1$) then x is the set of zeros of

$$p_k(t) + c_1p_{k-1}(t) + \cdots + c_{2k-1-n}p_{n+1-k}(t) \tag{3}$$

for some choice of $c_1,\cdots,c_{2k-1-n}$. Conversely, if x is the set of zeros of (3) (that is, (3) may be written as $\omega(t) = (t-x_1)\cdots(t-x_k)$) then

$$\int_{-1}^{1} \omega(t)\,p(t)\,\rho(t)\,dt = 0$$

for all $p \in P_{n-k}$, and given any $q \in P_n$, $q = \omega p + r$, $p \in P_{n-k}$, $r \in P_{k-1}$ with $q(x_i) = r(x)$, $i = 1,\cdots,k$. Thus

$$\int_{-1}^{1} q(t)\rho(t)dt = \int_{-1}^{1} r(t)\rho(t)dt = \int_{-1}^{1} L_k(q;t)\rho(t)dt$$
$$= \sum_{i=1}^{k} \left(\int_{-1}^{1} l_i(t)\rho(t)dt \right) q(x_i),$$

and we have an (n,k) qf provided that

$$\int_{-1}^{1} l_i(t)\rho(t)dt \neq 0, \; i = 1,\cdots,k. \tag{4}$$

Our interest in $(2k-3,k)$ qf leads to a study of the zeros of

$$q_k(t) = p_k(t) + c_1 p_{k-1}(t) + c_2 p_{k-2}(t),$$

which we write, for future convenience, in the form

$$q_k(t) = p_k(t) + (u_k - a)p_{k-1}(t) + (b + v_k)p_{k-2}(t). \tag{5}$$

In Section 2 we study these zeros and give a complete description of $(2k-3,k)$ qf in terms of (a,b). Section 3 is devoted to positive $(2k-3,k)$ qf and in Section 4 the qf for certain special weight functions are described in more detail.

2. The zeros of $q_k(t)$

In view of the three term recurrence formula we have

$$q_k(t) = (t-a)p_{k-1}(t) + bp_{k-2}(t). \tag{6}$$

If $\xi_1 < \xi_2 < \cdots < \xi_{k-1}$ are the zeros of p_{k-1} (they are in $(-1,1)$) then since $p_{k-2}(\xi_i) \neq 0$ we obviously have Lemma 1.

LEMMA. 1. *If $b = 0$ then the zeros of q_k are $\xi_1,\cdots,\xi_{k-1}$ and a. Conversely, if $q_k(\xi_i) = 0$ for some i then $b = 0$.*

As a consequence of Lemma 1, we make the following observation: if $b \neq 0$ every x which is a set of the zeros of (6) has a $(2k-3,k)$ qf based on it. That is, if $b \neq 0$ and $n = 2k-3$, (4) holds, for, suppose

$$\int_{-1}^{1} l_i(t)\rho(t)\,dt$$

is zero for $i = m$ and not zero for $i = j$ (there must be such a j). Then

$$0 = \int_{-1}^{1} p_{k-1}(t)\frac{q_k(t)}{(t-x_m)(t-x_j)}\rho(t)dt$$
$$= \frac{q_k'(x_j)}{x_j - x_m} p_{k-1}(x_j) \int_{-1}^{1} l_j(t)\rho(t)\,dt,$$

implies that $p_{k-1}(x_j) = 0$ and hence $b = 0$.

Conversely, if $b = 0$ and $a \in I$, $a \neq \xi_i$, $i = 1, \cdots, k-1$ there is no $(2k-3, k)$ qf based on $\{a, \xi_1, \cdots, \xi_{k-1}\}$ since if $\omega(t) = (t-a)p_{k-1}(t)$ and $l_i(t) = (p_{k-1}(t)/(\omega'(a))$ then

$$\int_{-1}^{1} l_i(t)\rho(t)dt = 0.$$

We may write (6) in the form

$$q_k(t) = p_{k-1}(t)\left((t-a) + b\frac{p_{k-2}(t)}{p_{k-1}(t)}\right) = p_{k-1}(t)r_k(t).$$

According to Lemma 1, if $b \neq 0$ the zeros of q_k coincide with those of r_k. Szegö [4, p. 47] showed that

$$Q(t) = \frac{p_{k-2}(t)}{p_{k-1}(t)} = \sum_{j=1}^{k-1} \frac{\lambda_j}{t - \xi_j};\ \lambda_j > 0,\ j = 1, \cdots, k-1. \tag{7}$$

Thus if I_0 denotes the interval $(-\infty, \xi_1), I_j\colon (\xi_j, \xi_{j+1})$, $j = 1, 2, \cdots, k-2$; I_{k-1}: (ξ_{k-1}, ∞) then

$$Q'(t) < 0,\ t \in I_j,\ j = 0, \cdots, k-1,$$

and $Q(t)$ is strictly monotone decreasing from ∞ to $-\infty$ in each $I_j, j = 1, \cdots, k-2$. Hence we have Lemma 2.

LEMMA 2. (i) *If $b \neq 0$ then q_k has at least one zero in each interval I_j, $j = 1, \cdots, k-2$; and if it has more than one zero in I_m, say, then it has three (counting multiplicities) there, and exactly one zero in each of I_j, $j \neq m$.*

(ii) *If $b < 0$ then $r_k(t)$ is strictly monotone increasing from $-\infty$ to ∞ in each I_j, $j = 0, \cdots, k-1$ and so $q_k(t)$ has k distinct real zeros.*

Moreover we can state Theorem 1.

THEOREM 1. *$q_k(t)$ has one zero in each of $[-1, \xi_1)$, $(\xi_{k-1}, 1]$, I_j, $j = 1, \cdots, k-2$, if, and only if,*

$$b < 0 \tag{8}$$

and

$$\begin{aligned} l_1(a, b) &= Q(1)b - a + 1 \geqq 0, \\ l_2(a, b) &= Q(-1)b - a - 1 \leqq 0. \end{aligned} \tag{9}$$

PROOF. (i) If (9) holds then $r_k(-1) \leqq 0$ and $r_k(1) \geqq 0$, hence (8) implies that $q_k(t)$ has the required zeros, in view of Lemma 2(ii).

(ii) Suppose $q_k(t)$ has a zero in each of the intervals $[-1, \xi_1)$, $(\xi_{k-1}, 1]$, I_j, $j = 1, \cdots, k-2$. Then $b \neq 0$ by Lemma 1. Suppose $b > 0$. Then we must have $r_k(-1) \geqq 0$ and $r_k(1) \leqq 0$; these inequalities imply $2 + b(Q(1) - Q(-1)) \leqq 0$, which, since $Q(1) > 0$ and $Q(-1) < 0$, is impossible. Thus we must have $b < 0$, hence $r_k(-1) \leqq 0$ and $r_k(1) \geqq 0$ must hold, and these inequalities are exactly (9).

Remark. The linear inequalities (8) and (9) are consistent. The set of (a, b) satisfying them form a triangle with the point $(u_k, -v_k)$ within it. If we denote the set by Δ, then the perimeter of the triangle, excluding the closed segment BC, is, of course, in Δ (see Fig. 1). This remark follows from the observations that $Q(1) > 0$, $Q(-1) < 0$, $v_k > 0$ and, as a consequence of the three term recurrence formula, $1 - u_k - v_k Q(1) > 0$ and $-1 - u_k - v_k Q(-1) < 0$.

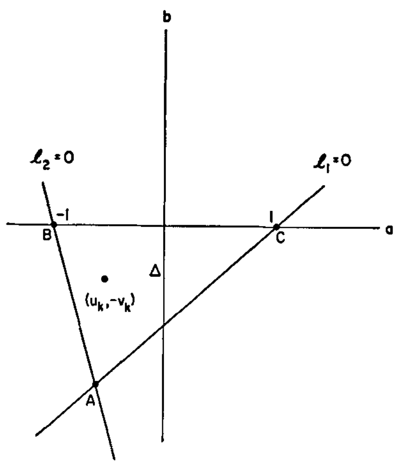

Fig. 1

Theorem 1 and Lemma 2 give a complete description of the location of the zeros of q_k when $b < 0$. We turn next to a detailed study of the set of (a, b) with $b > 0$ for which $(2k-3, k)$ qf exist.

If $t_1, t_2 \in I_j$, $j = 0, \cdots, k-1$ and $t_1 < t_2$, then t_1 and t_2 are zeros of $q_k(t)$ if, and only if, $r_k(t_1) = r_k(t_2) = 0$, which is equivalent to

$$b = \frac{t_1 - t_2}{Q(t_2) - Q(t_1)} \tag{10}$$

and

$$a = \frac{t_2 Q(t_1) - t_1 Q(t_2)}{Q(t_1) - Q(t_2)} = t_1 + bQ(t_1). \tag{11}$$

If $t_1 = t_2 = \tau$, we obtain

$$b = -\frac{1}{Q'(\tau)} \tag{12}$$

and

$$a = \tau - \frac{Q(\tau)}{Q'(\tau)}.$$

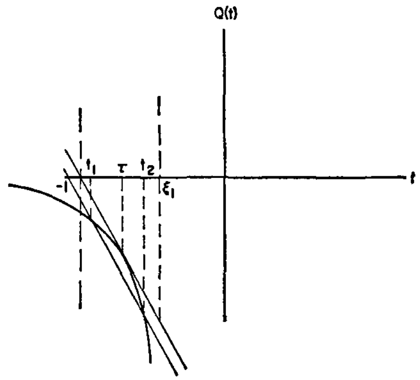

Fig. 2

Suppose now that $-1 \leqq t_1 < t_2 < \xi_1$. In view of Lemma 2 there is a $(2k-3, k)$ qf corresponding to each pair (t_1, t_2). Fig. 2 depicts $Q(t)$ in $[-1, \xi_1)$. If (t_1, t_2) gives b by (10) then the same b arises from infinitely many (t_1, t_2), namely, the abscissas of the intersections of all lines of slope $-1/b$ with $Q(t)$. Let $\tau(b)$ denote the unique value of t such that (12) holds. Then b arises from (t_1, t_2) with $-1 \leqq t_1 < \tau$, and b satisfies

$$0 < b < -\frac{1}{Q'(\tau)}. \tag{13}$$

To each b satisfying (13) there corresponds the a interval

$$-1 + bQ(-1) \leqq a < \tau + bQ(\tau) = \tau - \frac{Q(\tau)}{Q'(-1)}. \tag{14}$$

Thus (a, b) lies in the set S_0 delimited by the inequalities (13) and (14) (see Fig. 3). The curved boundary of S_0 is given by

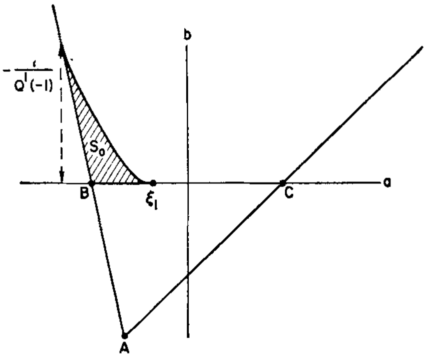

Fig. 3

$$a(\tau) = \tau - \frac{Q(\tau)}{Q'(\tau)}, \tag{15}$$

$$b(\tau) = -\frac{1}{Q'(\tau)}, \text{ for } -1 \leqq \tau < \xi_1 . \tag{16}$$

Thus

$$\frac{db}{da} = \frac{1}{Q} \tag{17}$$

and

$$\frac{d^2b}{da^2} = -\frac{(Q')^3}{Q^3Q''}, \tag{18}$$

which provide the qualitative features depicted in Fig. 3.

It is clear that each $(a, b) \in S_0$ leads to a pair of zeros (t_1, t_2), hence there is a one-to-one correspondence between S_0 and $(2k-3, k)$ qf having two nodes in $[-1, \xi_1)$.

Fix j, $1 \leqq j \leqq k-2$ and consider the interval I_j. $Q'(t) < 0$ and $Q''(t)$ is strictly monotone decreasing from ∞ to $-\infty$ there. Let τ_j be the unique zero of Q'' in I_j, so that $Q'(t)$ takes on once all values in $(-\infty, Q'(\tau_j)]$ in each of the intervals $(\xi_j, \tau_j]$ and $[\tau_j, \xi_{j+1})$. Every line which intersects $Q(t)$ in three distinct points of I_j must have slope $-1/b$ where

$$0 < b < -\frac{1}{Q'(\tau_j)}. \tag{19}$$

There are two lines of given slope $-1/b$, with b satisfying (19), which are tangent to $Q(t)$, one at $t = \underline{\tau}(b)$, the other at $t = \bar{\tau}(b)$, and

$$\xi_j < \underline{\tau} < \tau_j; \; \tau_j < \bar{\tau} < \xi_{j+1}.$$

Every line between these two tangents meets $Q(t)$ in three distinct points, with abscissas $t_1 < t_2 < t_3$. Thus to each b satisfying (19) there corresponds the a interval

$$\underline{\tau} + bQ(\underline{\tau}) < a < \bar{\tau} + bQ(\bar{\tau}) \tag{20}$$

in view of (11). Hence (a, b) lies in the set S_j defined by the inequalities (19) and (20). The curved boundary of S_j is given once again by the parametric equations (15) and (16) with $\xi_j \leqq \tau \leqq \xi_{j+1}$. It is clear that b attains its maximum at τ_j and, in view of (17), the boundary of S_j has a vertical tangent at γ_j, the zero of $Q(t)$ (hence of $p_{k-2}(t)$) in I_j. Depending on the relative sizes of τ_j and γ_j, S_j is either of type A, type B, or type C, as depicted in Fig. 4. Clearly each $(a, b) \in S_j$ yields a $(2k-3, k)$ qf with three nodes in I_j.

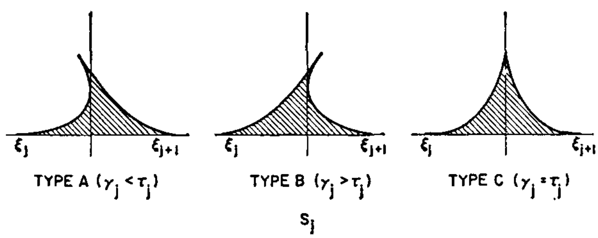

Fig. 4

The description of S_{k-1}, the set of (a, b) corresponding to $(2k-3, k)$ qf with two nodes in $(\xi_{k-1}, 1]$, resembles that of S_0 and we omit it.

This completes our description of all possible $(2k-3, k)$ qf. They correspond to all (a, b) which lie in Δ and a spiky crown atop Δ. We shall discuss these domains further when we examine some special weight functions in Section 4.

3. Positive $(2k - 3, k)$ quadrature formulae

We turn next to a consideration of positive $(2k-3, k)$ qf.

THEOREM 2. *A $(2k-3, k)$ qf is positive if, and only if, it is based on nodes which are the zeros of $q_k(t)$ with a, b satisfying* (8) *and* (9), *that is, it corresponds to some $(a, b) \in \Delta$.*

PROOF. (i) Suppose (1) is a positive $(2k-3, k)$ qf. Then

$$\sum_{i=1}^{k} w_i p_{k-1}(x_i) q(x_i) = 0 \tag{21}$$

for any $q \in P_{k-2}$. Since $w_i > 0$, $i = 1, \cdots, k$, this implies that the sequence $p_{k-1}(x_i)$, $i = 1, \cdots, k$ alternates in sign and thus $-1 \leqq x_1 < \xi_1 < x_2 < \cdots < \xi_{k-1} < x_k \leqq 1$. For, if $1 \leqq j \leqq k-1$, choose $q = q_j$ so that $q_j(x_i) = 0$, $i = 1, \cdots, k$; $i \neq j, j+1$ and $q_j(x_j) = 1$. Then $q_j(x_{j+1}) > 0$ and (21) becomes

$$w_j p_{k-1}(x_j) + w_{j+1} p_{k-1}(x_{j+1}) q_j(x_{j+1}) = 0,$$

which implies that either $p_{k-1}(x_j) p_{k-1}(x_{j+1}) < 0$ or $p_{k-1}(x_j) = p_{k-1}(x_{j+1}) = 0$. But if the latter case holds, then, if $m \neq j, j+1$ and we put $q = \bar{q}_m$ in (21), where $\bar{q}_m(x_m) = 1$ and $\bar{q}_m(x_i) = 0$, for $i \neq m, j, j+1$, we obtain $p_{k-1}(x_m) = 0$, and hence x does not consist of distinct nodes.

Theorem 1 now implies that $(a, b) \in \Delta$.

(ii) Suppose x is a set of zeros of $q_k(t)$ with $(a, b) \in \Delta$. The nodes $x_i(a, b)$ satisfy

$$-1 \leqq x_1(a, b) < \xi_1 < \cdots < \xi_{k-1} < x_k(a, b) \leqq 1$$

according to Theorem 1 and are continuous functions of (a, b). Thus, the $w_i(a, b)$ given by (2) are continuous functions of (a, b) for $(a, b) \in \Delta$. In view of the remarks following Lemma 1, $w_i(a, b) \neq 0$ for any i and $(a, b) \in \Delta$. Thus each $w_i(a, b)$ is of one sign throughout Δ, and since each $w_i(u_k, -v_k)$ is known to be positive (these being the weights for Gaussian quadrature), the sign is plus. This proves the theorem.

We can now summarize our description of $(2k-3, k)$ qf. They correspond in a one-to-one fashion to points (a, b) of $\Delta \cup S_0 \cup \cdots \cup S_{k-1}$. The positive qf correspond to Δ and all qf corresponding to $\cup S_k$, therefore, have one negative weight. The closed line segment of $b = 0$, which is a common boundary of Δ and $\cup S_k$, corresponds to the $(2k-3, k-1)$ Gaussian formula. All other boundary points of $\cup S_k$ correspond to a quadrature formula exact for P_{2k-3}, involving $k-1$ function evaluations and a derivative evaluation at one of the $k-1$ nodes. The cusp points correspond to quadrature formula, exact for P_{2k-3}, involving $k-2$ function evaluations and evaluations of the first two derivatives at one of the nodes. The $(2k-3, k)$ Gaussian formula corresponds to the point $(u_k, -v_k)$ of Δ, and all $(2k-2, k)$ qf correspond to the closed horizontal line segment passing through $(u_k, -v_k)$ and lying in Δ. Hence all $(2k-2, k)$ qf are positive.

Let us turn next to an examination of the nodes of positive $(2k-3,k)$ qf. One might be led to believe, by counting parameters in (1), that any two nodes can be assigned arbitrarily, subject only to the condition that the zeros of p_{k-1} interlace the set of nodes. We wish to show that this is not the case.

Suppose (8) and (9) hold and t_1, t_2 are the two arbitrary nodes. The choice of nodes $t_1 = -1$, $t_2 = 1$ is clearly possible. It corresponds to the lowest point of Δ, and the nodes of the quadrature formula are the abscissas of the points of intersection of the line joining $(-1, Q(-1))$ and $(1, Q(1))$ with $Q(t)$. (See Fig. 5.) Call these nodes λ_j, where $\lambda_j \in I_j$, $j = 1, \cdots, k-2$ and $\lambda_0 = -1$,

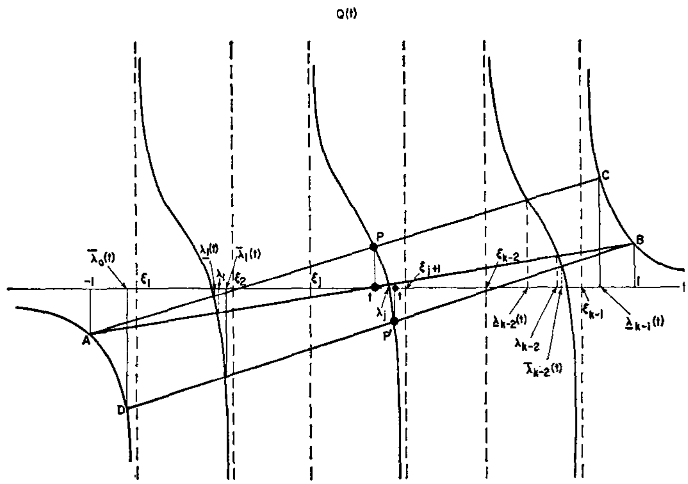

Fig. 5

$\lambda_{k-1} = 1$. Suppose next that $t_1 = -1$ and $t_2 = t \in I_j, (j = 1, \cdots, k-1)$ with $\xi_j < t \leqq \lambda_j$. The complete set of nodes is $\underline{\lambda}_i(t)$, $i = 0, \cdots, k-1$ where $\underline{\lambda}_0(t) = -1$, $\underline{\lambda}_j(t) = t$ and $\xi_i < \underline{\lambda}_i(t) \leqq \lambda_i$, $i = 1, \cdots, k-1$. Similarly if $t_1 = t \in I_j$, $t_2 = 1$ and $\lambda_j \leqq t < \xi_{j+1}$ we obtain a quadrature formula with nodes $\bar{\lambda}_i(t)$, $i = 0, \cdots k-1$, satisfying $\lambda_i \leqq \bar{\lambda}_i(t) < \xi_{i+1}$ for $i = 0, \cdots, k-2$ and $\bar{\lambda}_{k-1}(t) = 1$.

Now if

$$\xi_k < t_1 < \lambda_j$$

all quadrature formulae having t_1 as a node have nodes at the abscissas of the intersection with Q of all lines through $(t_1, Q(t_1))$, say the point P in Fig. 5, which have slopes at least as great as that of $\overline{AC}$. Therefore, t_1 can be coupled only with $t_2 \in I_i$ satisfying

$$\xi_i < t_2 \leqq \underline{\lambda}_i(t_1) \text{ for } i > j, \text{ and}$$

$$\underline{\lambda}_i(t_1) \leqq t_2 < \xi_{i+1} \text{ for } i < j.$$

Similarly, if

$$\lambda_j \leqq t_1 < \xi_{j+1},$$

t_1 can be coupled only with $t_2 \in I_i$ satisfying

$$\xi_i < t_2 \leqq \bar{\lambda}_i(t_1) \text{ for } i > j, \text{ and}$$

$$\bar{\lambda}_i(t_1) \leqq t_2 < \xi_{i+1} \text{ for } i < j.$$

In particular no t_1 satisfying $\xi_j < t_1 \leqq \lambda_j$ can be coupled with a t_2 satisfying $\lambda_i < t_2 < \xi_{i+1}$ if $i > j$. Thus certain pairings are prohibited, and we cannot assign two nodes arbitrarily.

4. Some special weight functions

The shape of the boundary curves of the spiky regions, S_k, is governed by the relative positions of γ_j and τ_j, the zeros of Q and Q'' respectively in the interval I_j. For some special choices of the weight function $\rho(t)$, we shall determine these relative positions.

First we observe that if $\rho(t)$ is an even function then $u_k = 0$ and $\Delta \cup S_0 \cup S_1 \cup \cdots \cup S_{k-1}$ is symmetric with respect to the b-axis. Suppose now that

$$\rho(t) = (1-t^2)^{\lambda-\frac{1}{2}}, \ \lambda > -\tfrac{1}{2}$$

so that the orthogonal polynomials, p_k, are the ultraspherical polynomials which satisfy

$$np_n = 2(n+\lambda-1)tp_{n-1} - (n+2\lambda-2)p_{n-2}, \quad n = 2,3,\cdots,$$

$$(1-t^2)p_n' = (n+2\lambda)tp_n - (n+1)p_{n+1},$$

and

$$(1-t^2)p_n'' = (2\lambda+1)tp_n' - n(n+2\lambda)p_n,$$

refer to Szegö [4, pp. 81, 83, 84]. Thus if $Q(t) = p_{k-2}(t)/p_{k-1}(t)$, application of the above equalities in a judicious manner yields

$$(1-t^2)Q''(t) = AtQ'(t) + BQ'(t)Q(t) + CQ(t)$$

where

$$A = (4k+4\lambda-3)(3-2\lambda-2k),$$

$$B = -2(k+2\lambda-2)(3-2\lambda-2k),$$

$$C = 2k+2\lambda-3.$$

Since $\lambda > -\frac{1}{2}$ and $k \geqq 2$ we obtain $A < 0$, and hence since $Q'(\gamma_j) < 0$

$$\operatorname{sgn} Q''(\gamma_j) = \operatorname{sgn} \gamma_j .$$

Thus if $\gamma_j < 0$ we have $\tau_j < \gamma_j$ and we have a type B spike (see Fig. 4), while if $\gamma_j = 0$ (which can occur only when i is odd) we see that $\tau_j = 0$ and we obtain a type C spike. The rest of the configuration follows from symmetry. In Fig. 6 we depict $\Delta \cup S_0 \cup S_1 \cup \cdots \cup s_4$ as computed in the Chebyshev case, that is $\rho(t) = (1-t^2)^{-\frac{1}{2}}$.

We conclude by mentioning the previous literature on the topic of this paper known to us. Fejér [1] showed that in the Legendre case ($\rho(t) \equiv 1$) if $q_k(t) = p_k(t) + c_1 p_{k-1}(t) + c_2 p_{k-2}(t)$ has n real distinct zeros in I and $c_2 \leqq 0$ then the weights in the quadrature formula based on these zeros are positive. (In a footnote he indicates that he intends to investigate the polynomials $q_k(t)$ further elsewhere, but examination of his collected works [2] does not reveal such an investigation). Our results show that the condition $c_2 \leqq 0$ can be relaxed somewhat.

P. Rabinowitz brought to our attention the work of Soulé [3] in which $(2k-3, k)$ qf are used as a control on Gaussian quadrature.

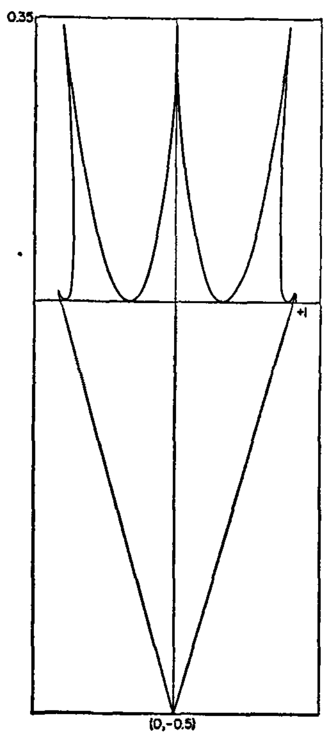

Fig. 6

References

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2. L. Fejér, *Gesammelte Arbeiten*, 2 vols., Birkhauser, Basel, 1970.

3. J. L. Soulé, *Formules de quadrature contrôlée*, Rapport No. 23, Dep't. de Calcul Electronique, Comm. à L'Energie Atomique, 1966.

4. G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc., New York, 1959.

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